Gravothermal Instabilities in dS and AdS

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Layout

Thermodynamics of self-gravitating gas

- Non-extensivity of energy and entropy
  due to the long range character of gravitational interaction
- Non-equivalence of ensembles
  Canonical and microcanonical ensembles lead to different predictions[1]
- Negative Heat Capacity (NHC) appears in the microcanonical ensemble[2]
  Virial theorem: $2K + U = 0 \Rightarrow E = -K \Rightarrow C_V = -\frac{3}{2}Nk$
  most astrophysical systems have NHC
  NHC region is replaced by phase transition in canonical ensemble
- energy decrease⇔shrinking⇔temperature increase
  energy increase⇔expansion⇔temperature decrease
  crucial for stellar stability[3]: Gravothermal vs Thermonuclear effects
- important for Quantum Gravity

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[1] T.Padmanabhan, Phys. Rep. 188, 285, (1990)
[2] D.Lynden-Bell, R.Wood, MNRAS 138, 495, (1968)
[3] H.A.Posch,W.Thirring, PRL 95, 251101 (2005)

Non-equivalence between microcanonical and canonical ensembles

$$\delta S \text{ with constant } E \Leftrightarrow \delta F \text{ with constant } T$$
$$\delta^2 S \neq \delta^2 F$$

This means that both ensembles give the same equilibria
but the onset of the instability changes.

The equation governing equilibria is obtained by two conditions

1. $\nabla^2 \phi = 4\pi G \rho$
2. $\delta S = 0$ with constant E, $M = mN$

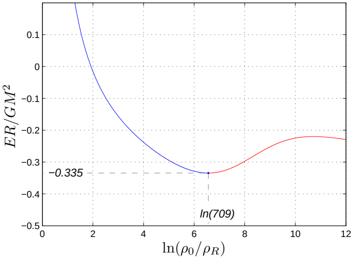
For $S = \int f \ln f d^3\vec{r} d^3\vec{\upsilon}$, where $f = f(\vec{r}, \vec{\upsilon})$ is the one particle distribution function in the mean field approximation, and assuming spherical symmetry we get

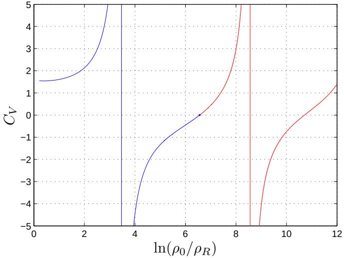
$$f = (\beta/2\pi)^{3/2} \rho_0 e^{-\beta(\phi - \phi(0))} e^{-\beta \upsilon^2/2}$$

that leads to the Emden equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d}{dr}\phi\right) = 4\pi G \rho_0 e^{-\beta(\phi - \phi(0))}$$

- Microcanonical ensemble (E const.): Gravothermal Catastrophe[4,5]
- Canonical ensemble (T const.): Isothermal Collapse[6]

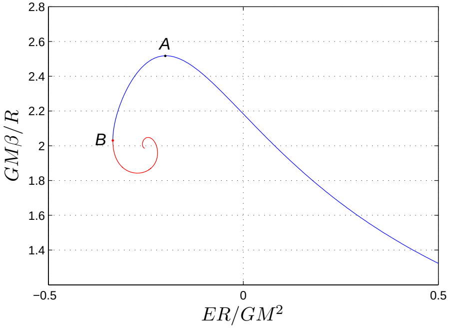




Poincaré's theorem: stability changes at an energy (microcanonical) or temperature (canonical) extremum[7,8] in the series of equilibria

[4] V.A. Antonov, Vest. Leningrad Univ. 7, 135, (1962)
[5] D. Lynden-Bell and R. Wood, MNRAS 138, 495, (1968)
[6] P.H. Chavanis, A& A 381, 340, (2002)
[7] J. Katz, MNRAS 183, 765, (1978)
[8] H. Poincaré, Acta. Math. 7, 259, (1885)

The series of equilibria $\beta = \beta(E)$.



- Microcanonical ensemble:
  an instability sets in at point B; no equilibria exist for
  $E < E_B \Leftrightarrow R > R_B$
- Canonical ensemble:
  an instability sets in at point A; no equilibria exist for
  $T < T_A \Leftrightarrow R < R_A$

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Adding a Cosmological Constant[9,10]

The Emden equation becomes

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\phi\right) = 4\pi G\rho_0 e^{-\beta(\phi-\phi(0))} - 8\pi G\rho_\Lambda \quad , \ \rho_\Lambda = \frac{\Lambda c^2}{8\pi G}$$

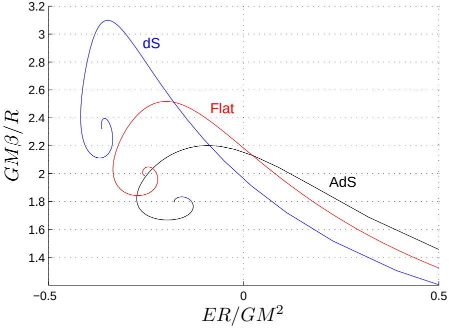
Dimensionless variables: $x = r\sqrt{4\pi G\rho_0\beta}$, $y = \beta(\phi - \phi(0))$, $\lambda = \frac{2\rho_\Lambda}{\rho_0}$

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{d}{dx}y\right) = e^{-y} - \lambda \quad , \ y(0) = y'(0) = 0$$

Dimensionless temperature, energy and mass:

$$\bar{\beta} = \frac{GM\beta}{R} \quad , \quad Q = \frac{ER}{GM^2} \quad , \quad m = \frac{M}{2M_\Lambda}$$

[9] M.Axenides, G. Georgiou, Z.R., [arXiv:1206.2839], to appear in Phys. Rev. D
[10] M.Axenides, G. Georgiou, Z.R., to appear in J. Phys. Conf. Ser.

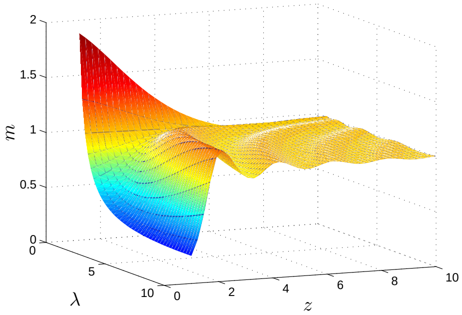


As the cosmological constant increases, there exist equilibrium states at even lower temperatures and energies.

For $z = x(R)$ we get $m = 3\bar{\beta}/\lambda z^2$. We want $\bar{\beta}(z)$ and $Q(z)$.
Have to solve Emden-Λ for various z keeping M, i.e. m fixed.
We constructed a computer code that solves Emden-Λ for various z keeping m constant. Then $\bar{\beta}$, Q are found by the equations:

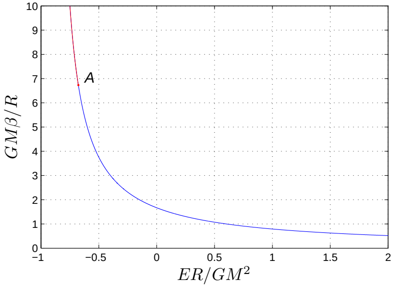
$$\bar{\beta}(z) = zy'(z) + \frac{1}{3}\lambda z^2 \, , \; Q(z) = \frac{z^2 e^{-y(z)}}{\bar{\beta}^2} - \frac{3}{2\bar{\beta}} - \frac{\lambda}{2\bar{\beta}^2 z}\int_0^z x^4 e^{-y(x)} dx$$

For $m = const.$ there are various series of solutions corresponding to pairs λ, z.
For $m = 1$ ($M = 2M_\Lambda$) there are infinite series.
$m = \frac{3}{8\pi}\frac{M}{\rho_\Lambda R^3}$



At $R_H = (\frac{3M}{8\pi\rho_\Lambda})^{\frac{1}{3}}$ there exists a homogeneous solution $\rho = 2\rho_\Lambda = const.$, that is the non-relativistic analogue of Einstein's static Universe.
The homogeneous solution suffers a transition to instability at $T_{cr} = GM/6.73R_H$.

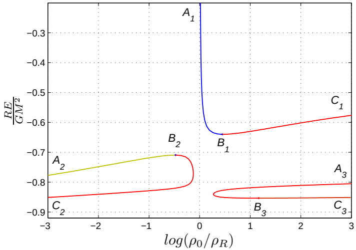
$R = R_H$



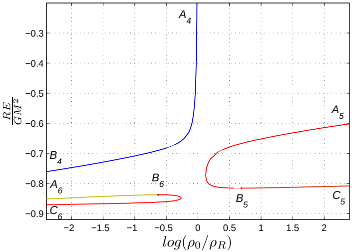
At $R = R_H$ there exist infinite other solutions. All suffer a transition from stability ($\rho_0 < \rho_R$) to instability ($\rho_0 > \rho_R$) at $\rho_0 = \rho_R$.

Microcanonical ensemble

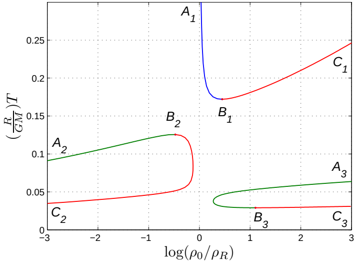
$R < R_H$

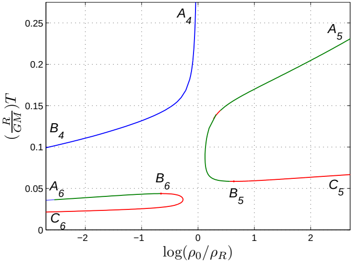


$R > R_H$



Canonical ensemble





Second Variation of Entropy

$$\delta^2 S = \int_0^R \int_0^R \delta M(r_2)\hat{K}(r_1, r_2)\delta M(r_1) dr_1 dr_2$$

where δM is a local mass perturbation and:

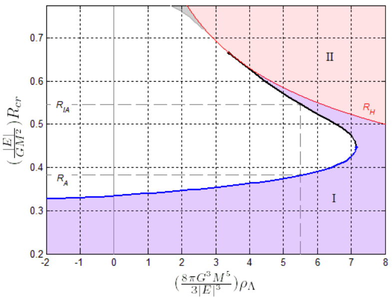
$$\hat{K}(r_1, r_2) = -\frac{\phi'(r_1)\phi'(r_2)}{3MT^2} + \frac{1}{2}\delta(r_1 - r_2)\left[\frac{G}{Tr_1^2} + \frac{d}{dr_1}\left(\frac{1}{4\pi\rho r_1^2}\frac{d}{dr_1}\right)\right]$$

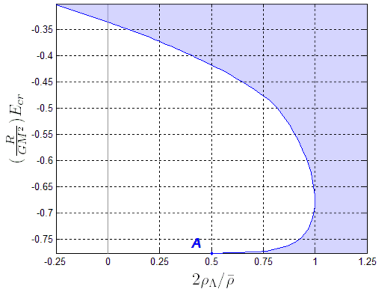
The sign of $\delta^2 S$ is determined by the sign of the eigenvalues ξ of the eigenvalue problem:

$$\int_0^R \hat{K}(r, r_1)F_\xi(r_1)dr_1 = \xi F_\xi(r) \qquad (1)$$

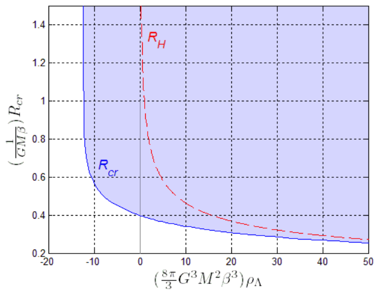
with $F_\xi(0) = F_\xi(R) = 0$. We developed an algorithm that can numerically determine eigenvalues and eigenstates (the perturbations) of equation (1).

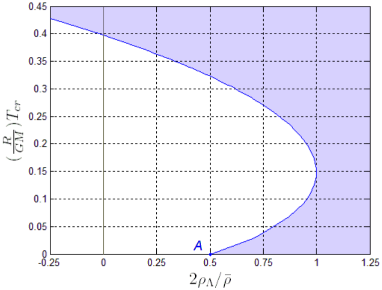
Microcanonical Ensemble





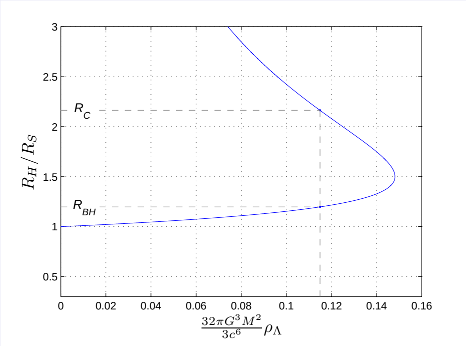
Canonical Ensemble





Schwartzscild-dS

$$1 - \frac{2GM}{c^2 R_H} - \frac{8\pi G}{3c^2}\rho_\Lambda R_H^2 = 0$$



Newtonian stable region corresponds to relativistic unstable region.
Explanation (without Λ) in: P.H. Chavanis, A&A, **381**, 709 (2002)

Layout

Tolman-Oppenheimer-Volkov (TOV) with Λ

Emden $\Leftrightarrow$ TOV for dust [Chandrasekhar (1972)][11]
For spherically symmetric and static systems:

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2 d\Omega\ ,\ \ T_{\mu\nu} = pg_{\mu\nu} + (p + \rho c^2)U_\mu U_\nu$$

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \Rightarrow \begin{cases} \frac{dp}{dr} = -\frac{(p/c^2+\rho)\left(\frac{GM(r)}{r^2}+4\pi Gpr/c^2-\frac{8\pi G}{3}\rho_\Lambda r\right)}{\left(1-\frac{2GM(r)}{rc^2}-\frac{8\pi G}{3c^2}\rho_\Lambda r^2\right)} \\ \frac{dM(r)}{dr} = 4\pi\rho r^2 \end{cases}$$

TOV expresses the condition for Hydrostatic equilibrium.
Dust $\Leftrightarrow p/c^2 \to 0$ gives:

$$\frac{1}{\rho}\frac{dp}{dr} = -\frac{GM(r)}{r^2} - \frac{8\pi G}{3}\rho_\Lambda r \quad \overset{p=\rho/\beta}{\Longrightarrow} \quad \text{Emden-}\Lambda$$

[11] S. Chandrasekhar, 'General Relativity' p. 185-199, Oxford (1972)

TOV from Thermodynamics[13, 14]

Assumptions:

- Spherical Symmetry and Staticity
- 1^{st} Law of Thermodynamics (locally): $ds = \frac{1}{T} d\rho - \frac{\mu}{T} dn$
- Gibbs-Duhem relation (locally): $Ts = \rho + p - \mu n$
- An expression for the space Volume Element

$$dV = \left(1 - \frac{2GM(r)}{r} - \frac{\Lambda}{3} r^2\right)^{-\frac{1}{2}} 4\pi r^2 dr \ , \ \frac{dM(r)}{dr} = 4\pi \rho r^2$$

Variation of the entropy $S = \int_0^R s dV$ under the constraint $N = \int_0^R n dV = const.$ gives the TOV-Λ equation!
Looks as if there is a 'Thermodynamic sector' in General Relativity[12].

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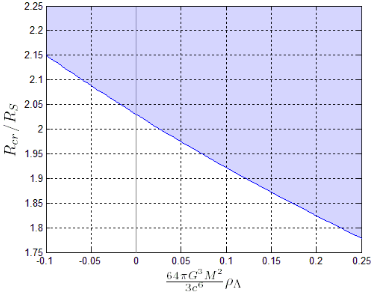
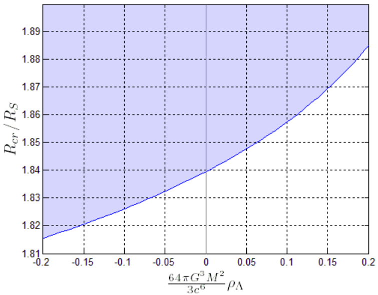
[12] J.Oppenheim, 'Thermodynamic Sector in Quantum Gravity', [arXiv:gr-qc/0112001]
[13] R.Sorkin, R.M.Wald, Z.Z.Jiu, Gen. Rel. Grav., 13, 1127 (1981)
[14] S.Gao, Phys.Rev.D 84, 104023 (2011)

Assume an equation of state of the form $p = w\rho$

Radiation $w = 1/3$



Stiff matter $w = 1$



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Results

- dS ($\Lambda > 0$) tends to stabilize the system.
- AdS ($\Lambda < 0$) tends to destabilize the system.
- Re-entrant phase transitions in dS.
- Microcanonical and Canonical ensembles are different.
  Microcanonical: Instability for $R > R_{cr}$
  Canonical: Instability for $R < R_{cr}$
- Similarity between Newtonian microcanonical ensemble and Schwartzschild-dS.
- General relativistic system behaves like the Newtonian in the canonical ensemble.

Thank you!