# Nonlinear Field Theory with Topological Solitons: Skyrme Models

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# Solitons

Seen firstly by J. Scott Russel in the Edinburgh canal as a solitary wave (1834).

- They conserve the form.
- They are localised solutions.
- After collisions with other solitons they emerge unchanged.

## Topological Solitons

- Collective excitations of the relevant degrees of freedom.
- Existence and stability caused by the global topological structure of the base and field space.

# Skyrme

**T.H.R. Skyrme:** Low-energy field theory of QCD.

- 3+1 dimensional nonlinear field theory.
- Little brother: Baby Skyrme model (2+1)
- Primary fields in the Skyrme model are the pions.
- Nucleons and nuclei are described by collective nonlinear excitations of the fundamental degrees of freedom.
- Topological charge identified with the baryon number.
- Derrick theorem $\Rightarrow$ Terms quadratic and quartic in first derivatives.
- Generalizations: Potential term and sextic term

The more general model:

$$\mathcal{L}_{Skr} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \mathcal{L}_0$$

$$\mathcal{L}_2 = -\frac{f_\pi^2}{4}\mathrm{Tr}(L_\mu L^\mu) \qquad \mathcal{L}_4 = -\frac{1}{32e^2}\mathrm{Tr}\left([L_\mu, L_\nu]^2\right)$$

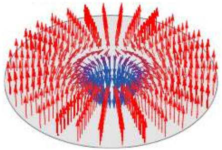
$$\mathcal{L}_6 = \lambda^2\pi^4 B_\mu^2 \qquad \mathcal{L}_0 = -\mu^2 V$$

where $U \in SU(2)$ and

$$L_\mu = U^\dagger \partial_\mu U$$

$$B^\mu = \frac{1}{24\pi^2}\mathrm{Tr}(\epsilon^{\mu\nu\rho\sigma} U^\dagger \partial_\nu U U^\dagger \partial_\rho U U^\dagger \partial_\sigma U).$$

- Not only applications in Particle Physics: Condensed Matter (Ferromagnetic materials)



# BPS Bound

**BPS:** Bogomol'nyi, Prasad and Sommerfield.

- Systems with a topological lower energy bound.
- BPS equations for solitons saturating the bound.
- Simplification on field equations going from second to first order.
- Stable BPS states since they minimize the energy.
- Interesting energy bound for the Skyrme Model: $E \propto n$.
- An example: BPS Skyrme Model (C.Adam, J. Sanchez-Guillen, A. Wereszczynski, Phys. Lett. B **691,** 105 (2010)):

$$\mathscr{L}_{BPS} = \mathscr{L}_6 + \mathscr{L}_0$$

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# The model

$$\mathscr{L} = -\frac{\lambda^2}{4}(D_\alpha\vec{\phi} \times D_\beta\vec{\phi})^2 - \mu^2 V(\vec{n}\cdot\vec{\phi}) - \frac{1}{4g^2}F_{\alpha\beta}^2$$

- Vector of scalar fields

$$\vec{\phi} = (\phi_1, \phi_2, \phi_3) \qquad \vec{\phi}^2 = 1.$$

- Covariant derivative

$$D_\alpha\vec{\phi} = \partial_\alpha\vec{\phi} + A_\alpha\vec{n}\times\vec{\phi}.$$

## Winding number or topological degree

Since for finite energy field configurations $\vec{\phi}: S^2 \to S^2$

$$\deg[\vec{\phi}] = \frac{1}{4\pi}\int d^2x\,\vec{\phi}\cdot\partial_1\vec{\phi}\times\partial_2\vec{\phi} = k, \qquad k\in\mathbb{Z}.$$

# Standard static ansatz

We assume $\vec{n} = (0,0,1)$ and

$$\vec{\phi}(r,\varphi) = \begin{pmatrix} \sin f(r) \cos n\varphi \\ \sin f(r) \sin n\varphi \\ \cos f(r) \end{pmatrix}$$

$$A_0 = A_r = 0, \qquad A_\varphi = na(r)$$

So the electric and magnetic fields

$$E_i = 0 \qquad B = \partial_1 A_2 - \partial_2 A_1 = \frac{na'(r)}{r}$$

whereas the winding number is just $\quad \mathrm{deg}[\vec{\phi}] = n$

and the old potential $\quad V_o = 1 - \phi_3$

# Standard static ansatz

For simplicity we introduce

$$y = r^2/2 \qquad h = \frac{1}{2}(1 - \cos f)$$

## Boundary conditions

$$h(0) = 1 \Leftrightarrow f(0) = \pi \qquad a(0) = 0$$

$$a_y(y = y_0) = 0 \qquad h(y_0) = h_y(y_0) = 0$$

or

$$\lim_{y \to \infty} h(y) = 0 \qquad \lim_{y \to \infty} a_y(y) = 0$$

# The BPS bound

After non-trivial manipulation

$$E \geq E_0\lambda^2 \int d^2xqW' = 4\pi|n|E_0\lambda^2\langle W'\rangle_{S^2} = 2\pi|n|E_0\lambda^2|W(-1)|$$

$n = \deg[\vec{\phi}]$

and the BPS equations

$$Q = W'$$

$$B = -g^2\lambda^2 W$$

where

$$Q \equiv \vec{\phi}\cdot D_1\vec{\phi}\times D_2\vec{\phi} = q + \epsilon_{ij}A_i\partial_j(\vec{n}\cdot\vec{\phi})$$

# Superpotential equation

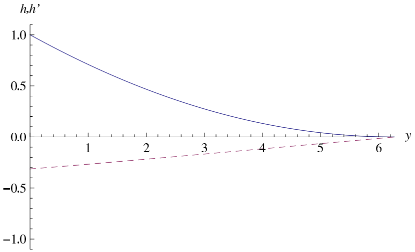
## Superpotential $W$ and superpotential equation

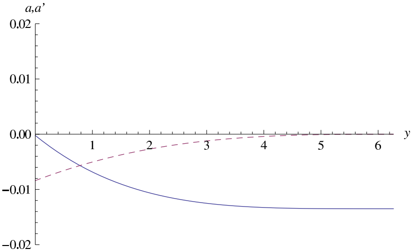
$$\lambda^2 W'^{\,2} + g^2 \lambda^4 W^2 = 2\mu^2 V$$

- Very similar to fake supergravity!!
- Self-gravitating domain walls $\Rightarrow$ Terms with different signs.
- Extremal, supersymmetric black holes $\Rightarrow$ They enter with the same sign.

# Numerical solutions

Solutions for $\mu^2 = 0.1$ and $g = 0.1$



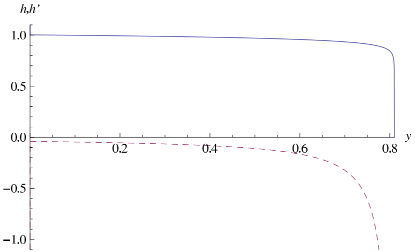



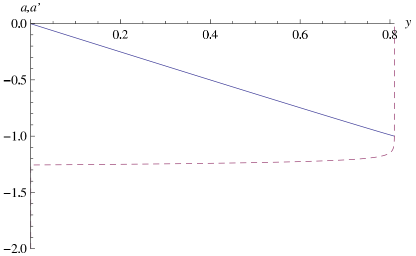

## Magnetic flux

$$\Phi = \int r dr d\varphi B = 2\pi n \int dy a_y = 2\pi n a(y_0)$$

# Numerical solutions

Solutions for $\mu^2 = 0.1$ and $g = 2$



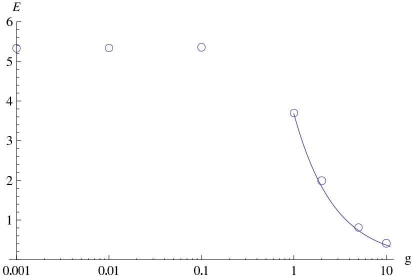



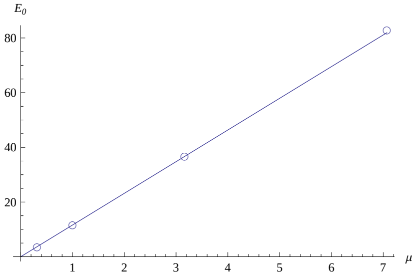

## Magnetic flux

$$\Phi = \int r dr d\varphi B = 2\pi n \int dy a_y = 2\pi n a(y_0)$$

# Numerical solutions

Energy as function of the parameters $g$ and $\mu$ respectively



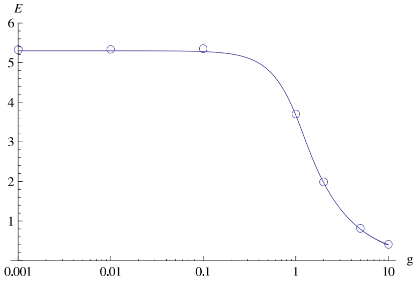




We have chosen $\mu = 0.1$ in the first case and $g = 0.1$ in the second one

# Numerical solutions

Comparing energy with the BPS bound

$$E_B = 2\pi n \lambda^2 |W(h=1)|$$




C.Adam, C.Naya, J. Sanchez-Guillen, A. Wereszczynski, Phys. Rev. D **86,** 045010 (2012)

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# Vector models

- How solitons interact with lightest fields?
- Two posibilities:
  - $U(1)$ Maxwell field: Model above.
  - Vector mesons: Here we go.
- After pions, omegas are the lightest mesons.
- Coupling with solitons: $\omega_\mu B^\mu$.
- Be aware of changes in the typical mass hierarchy.

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# The model

$$\mathscr{L} = -\mu^2 V(\phi_3) - \frac{1}{4}(\partial_\mu \omega_\nu - \partial_\nu \omega_\mu)^2 + \frac{1}{2} M^2 \omega_\mu^2 + \lambda' \omega_\mu B^\mu$$

- Topological current

$$B^\mu = -\frac{1}{8\pi} \epsilon^{\mu\alpha\beta} \, \vec{\phi} \cdot (\partial_\alpha \vec{\phi} \times \partial_\beta \vec{\phi})$$

- Generalized old baby potentials

$$V = \left( \frac{1 - \phi_3}{2} \right)^\alpha \qquad \alpha \geq 1$$

# Ansatz

Stereographic projection

$$\vec{\phi} = \frac{1}{1+|u|^2}(u+\bar{u}, -i(u-\bar{u}), |u|^2-1)$$

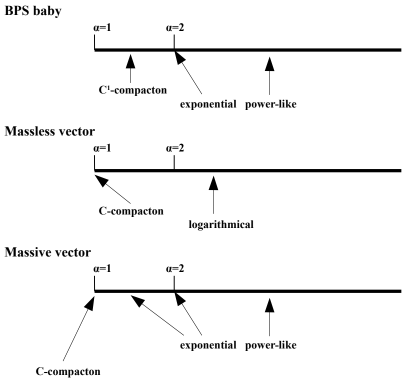
Axially symmetric ansatz:

$$u = f(r)e^{in\phi} \qquad \omega_0 \equiv \omega = \omega(r) \qquad \omega_i = 0$$

## Boundary conditions

$$f(r=0) = \infty \qquad f(r=R) = 0$$
$$\omega'(r=0) = 0 \qquad \omega(r=R) = 0$$
$$f'(r=R) = 0 \qquad \omega'(r=R) = 0$$

# Comparing results



# BPS case: $\alpha = 2$

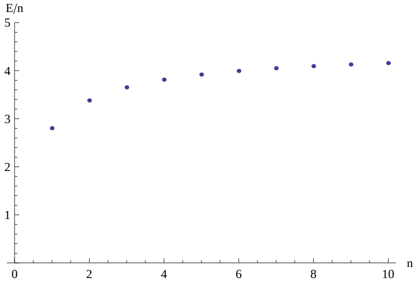
Solution

$$g(y) = \frac{1}{K_1(2/\beta)} \frac{K_1(\frac{2}{\beta}\sqrt{1+\beta y})}{\sqrt{1+\beta y}} \qquad \omega_y = -\sqrt{2}\frac{\mu}{M}g$$

with $K_1$ the modified Bessel function of the second type and $\beta = \frac{2\mu}{n\lambda M}$.

BPS energy bound:

$$E = -\frac{\pi}{2}\left(\frac{n\lambda}{\mu}\right)^2 \omega(0)\omega_x(0)$$
$$\simeq \sqrt{2}\pi\frac{n\lambda\mu}{M}\left(1 - \frac{1}{4}\beta + \cdots\right)$$



# Infinite mass limit

Assuming that $M \to \infty$

$$\mathscr{L} = -\mu^2 V(\phi_3) + \frac{1}{2} M^2 \omega_\mu^2 + \lambda' \omega_\mu B^\mu$$

Field equation for $\omega$ mesons

$$\omega_\mu = -\frac{\lambda'}{M^2} B_\mu$$

$$\Rightarrow \quad \mathscr{L} = -\mu^2 V(\phi_3) - \frac{1}{2} \frac{\lambda'^{\,2}}{M^2} B_\mu^2$$

BPS baby Skyrme model!!

C.Adam, C.Naya, J. Sanchez-Guillen, A. Wereszczynski, Phys. Rev. D **86,** 045015 (2012)

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# The model

$$\mathscr{L} = -\mu^2 V(U, U^\dagger) - \frac{1}{4}(\partial_\mu \omega_\nu - \partial_\nu \omega_\mu)^2 + \frac{1}{2} M^2 \omega_\mu^2 + \lambda' \omega_\mu B^\mu$$

- Topological current

$$B^\mu = \frac{1}{24\pi^2} \mathrm{Tr}(\epsilon^{\mu\nu\rho\sigma} U^\dagger \partial_\nu U U^\dagger \partial_\rho U U^\dagger \partial_\sigma U)$$

- Generalized Skyrme potentials

$$V = \left( \frac{1 - \mathrm{Tr} U}{2} \right)^\alpha = (1 - \cos\, \xi)^\alpha$$

# Static ansatz

Parametrizations

$$U = e^{i\xi\vec{n}\cdot\vec{\tau}} = \cos\xi + i\sin\xi\,\vec{n}\cdot\vec{\tau}$$

$$\vec{n} = \frac{1}{1+|u|^2}(u+\bar{u}, -i(u-\bar{u}), 1-|u|^2)$$

Assuming the static ansatz

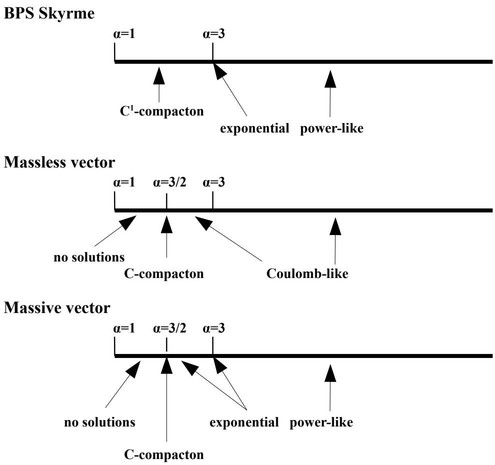
$$\omega_0 \equiv \omega = \omega(r) \qquad \omega_i = 0 \qquad \xi = \xi(r) \qquad u = \tan\frac{\theta}{2}e^{in\phi}$$

## Boundary conditions

$$\xi(r=0) = \pi \qquad \omega(r=R_0) = 0$$

$$\omega_r(r=0) = 0 \qquad \omega(r=R_0) = 0$$

# Comparing results



# BPS potential

Potential allowing BPS solitons

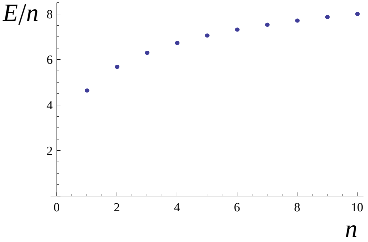
$$V_{BPS} = \frac{1}{4}(\xi - \cos\xi \sin\xi)^2 \quad \Rightarrow \quad E = \frac{\pi^2 n^2 \lambda^2}{2}\omega(0)$$

- Massless case

$$E = \frac{\pi^4}{8\sqrt{2\sqrt{2}}}\lambda\sqrt{\frac{\lambda}{\mu}}n^{3/2}$$

- Finite mass

$$E \sim n^{6/5}$$




C.Adam, C.Naya, J. Sanchez-Guillen, A. Wereszczynski, Phys. Rev. D **86,** 085001 (2012)

# Conclusions

- Three different versions of the Skyrme Model were presented.
- Couplings to a gauge and a vector meson field.
- For each model BPS bounds have been discovered.
- Superpotential equation similar to that in fake supergravity.
- In the gauged model, BPS soliton exists for monotonically growing potentials.
- For some potentials, the vector BPS models differ qualitatively from the BPS ones.
- Important difference for the standard pion mass potential.


Thank you!