A New Fluid Description of a Radiating Plasma

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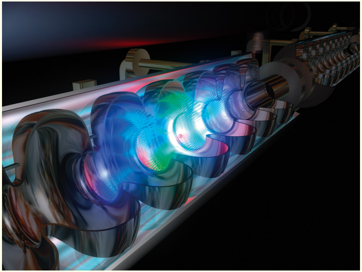

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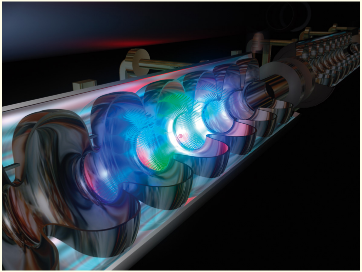

- As a charged particle is accelerated it emits electromagnetic radiation.
- Due to the charged nature of the particle it interacts with its own radiation field.
- To date, a detailed description of the effect this has on single particle motion has not been necessary.
- In modern accelerators the rate of energy loss due to this interaction is small and can be averaged out.

- However, accelerators due to be built in the near future such as ELI - The Extreme Light Infrastructure - are expected to operate with intensities of order 10^{23}W cm^{-2}, at which point the radiation reaction force becomes comparable to the Lorentz force!

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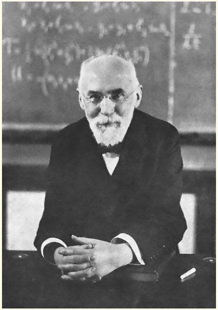
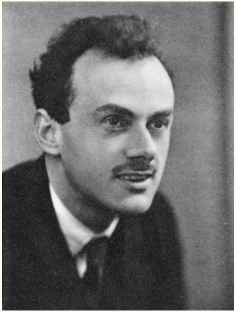
- Despite a considerable amount of research, exactly how radiation reaction will effect a single particle is still a mystery.

(a) Max Abraham



(b) Hendrik Lorentz



(c) Paul Dirac

- Lorentz and Abraham first attempted to describe this effect in 1904, and their result was later generalised to the relativistic regime by Dirac,

$$\dot{u}^a = -\frac{q}{m} F^a{}_b u^b + \tau \Pi^a{}_b \ddot{u}^b \tag{1}$$

Where $\Pi^a{}_b = \delta^a_b + u^a u_b$ is the projection orthogonal to the 4-velocity u^a.

$$\dot{u}^a = -\frac{q}{m} F^a{}_b u^b + \tau \Pi^a{}_b \ddot{u}^b$$

- The above equation has proved highly controversial.
- Solutions often involve exponentially growing proper acceleration or acausal behaviour.

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- There exists an alternate description, proposed by Landau and Liftshitz,

$$\dot{u}^a = -\frac{q}{m} F^a{}_b u^b - \frac{q}{m}\tau \left(\partial_d F^a{}_b u^b - \frac{q}{m} \Pi^a{}_b F^b{}_c F^c{}_d \right) u^d \qquad (2)$$

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- Obtained by iteration and truncation of the ALD equation.
- Intended to be an approximation to the ALD equation for low energies, but now thought by some to be as accurate.

A Kinetic Theory

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- Modern laser facilities accelerate electron bunches containing $\sim 10^8$ particles.
- Therefore radiation reaction is unlikely ever to be detected for a single particle.
- A more appropriate description of radiation reaction then, is not the one particle equation of motion but rather a kinetic theory applied to the bunch.
- A kinetic theory based on (2) exists but it is not obvious that such a theory would agree with one derived from the ALD equation.
- Hence, a fluid description based on the ALD equation is worthwhile.

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$$\mathcal{L}_L(f\omega) = 0 \tag{3}$$

where f is the particle density, ω is an infinitesimal area of phase-space and

$$L = \dot{x}^a \frac{\partial}{\partial x^a} + a^\mu \frac{\partial}{\partial v^a} + \left(\ddot{x}^a \ddot{x}_a v^\mu + \tau^{-1} \left(a^\mu + \frac{q}{m} F^\mu{}_a \dot{x}^a\right)\right) \frac{\partial}{\partial a^\mu}$$

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- Define the (l, n) natural moments of the distribution as

$$S^{a_1 \ldots a_l : b_1 \ldots b_n} = \int \dot{x}^{a_1} \ldots \dot{x}^{a_l} \ddot{x}^{b_1} \ldots \ddot{x}^{b_n} f \Omega \tag{4}$$

where $\Omega = \left(\dot{x}^0\right)^{-2} d^3v \wedge d^3a$ is the invariant measure

Equations of motion

- Using (4) and integrating the Vlasov equation yields

$$\partial_a S^{a:\emptyset} = 0 \tag{5}$$

$$\partial_a S^{ab:\emptyset} - S^{\emptyset:b} = 0 \tag{6}$$

$$\partial_a S^{a:b} - S^{b:c}{}_c - \tau^{-1}\left(S^{\emptyset:b} + \frac{q}{m}F^b{}_c S^{c:\emptyset}\right) = 0 \tag{7}$$

$$\partial_a S^{abc:\emptyset} - S^{b:c} - S^{c:b} = 0 \tag{8}$$

$$\partial_a S^{ab:c} - S^{\emptyset:bc} - S^{bc:d}{}_d - \tau^{-1}\left(S^{b:c} + \frac{q}{m}F^c{}_d S^{bd:\emptyset}\right) = 0 \tag{9}$$

$$\partial_a S^{a:bc} - S^{b:cd}{}_d - \tau^{-1}\left(2S^{\emptyset:bc} + \frac{q}{m}F^b{}_d S^{d:c} + \frac{q}{m}F^c{}_d S^{d:b}\right) = 0 \tag{10}$$

- Introduce the centred moments $R^{a_1 \ldots a_l : b_1 \ldots b_n}$, defined as

$$R^{a_1 \ldots a_l : b_1 \ldots b_l} \equiv \int (\dot{x}^{a_1} - U^{a_l}) \ldots (\dot{x}^{a_l} - U^{a_l}) \times \left(\ddot{x}^{b_1} - A^{b_1}\right) \ldots \left(\ddot{x}^{b_l} - A^{b_l}\right) \times f\Omega \quad (11)$$

and setting all those of third order and higher to zero generates the below *constraints*;

$$S^{a:\emptyset} = S^{\emptyset} U^{a} \qquad S^{a}{}_{a}{}^{:\emptyset} = -S^{\emptyset} \qquad S^{a:}{}_{a} = 0$$

$$S^{ab:}{}_{b}{}^{\emptyset} \qquad S^{a:}{}_{ab} = 0 \qquad S^{a}{}_{a}{}^{:b} = -S^{\emptyset:b}$$

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- Caution! We now have an over-prescribed system.

- The natural moments can be written in terms of the centred moments as shown below

$$S^{a:\emptyset} = S^{\emptyset} U^{a} \qquad S^{\emptyset:a} = S^{\emptyset} A^{a}$$

$$R^{ab:\emptyset} = S^{ab:\emptyset} - S^{\emptyset} U^{a} U^{b} \qquad R^{a:b} = S^{a:b} - S^{\emptyset} U^{a} A^{b}$$

$$R^{\emptyset:ab} = S^{\emptyset:ab} - S^{\emptyset} A^{a} A^{b}$$

- The system of equations (5) - (10) is too complicated to solve outright,
- Instead, expand each parameter as a formal power series in ϵ. For example;

$$S^{\emptyset:b} = S^{\emptyset:b}_{(0)} + \epsilon S^{\emptyset:b}_{(1)} + \epsilon^2 S^{\emptyset:b}_{(2)} + \cdots$$

$$S^{a:b} = S^{a:b}_{(0)} + \epsilon S^{a:b}_{(1)} + \epsilon^2 S^{a:b}_{(2)} + \cdots$$

$$F^{ab} = F^{ab}_{(0)} + \epsilon F^{ab}_{(1)} + \epsilon^2 F^{ab}_{(2)} + \cdots$$

- Note that the zeroth order terms represent the moments of the plasma when it is at equilibrium, the other terms represent higher order corrections.
- The above are then substituted into (5) - (10), and we collect like coefficients of ϵ.

- To zeroth order, i.e. all those terms that are not multiplied by ϵ, the system of equations is

$$S_{(0)}^{\emptyset:b} = 0 \tag{12}$$

$$-S_{(0)}^{b:c}{}_{c} - \tau^{-1}\left(S_{(0)}^{\emptyset:b} + \frac{q}{m}F_{(0)}^{b}{}_{c}S_{(0)}^{c:\emptyset}\right) = 0 \tag{13}$$

$$S_{(0)}^{b:c} = -S_{(0)}^{c:b} \tag{14}$$

$$-S_{(0)}^{\emptyset:bc} - S_{(0)}^{bc:d}{}_{d} - \tau^{-1}\left(S_{(0)}^{b:c} + \frac{q}{m}F_{(0)}^{c}{}_{d}S_{(0)}^{bd:\emptyset}\right) = 0 \tag{15}$$

$$-S_{(0)}^{b:cd}{}_{d} - \tau^{-1}\left(2S_{(0)}^{\emptyset:bc} + \frac{q}{m}F_{(0)}^{b}{}_{d}S_{(0)}^{d:c} + \frac{q}{m}F_{(0)}^{c}{}_{d}S_{(0)}^{d:b}\right) = 0 \tag{16}$$

- Note that since the zeroth order terms represent the equilibrium state, their derivatives vanish.

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- After some algebraic manipulation one finds that

$$\begin{aligned}\frac{q}{2m} \quad (\quad & F^b_{(0)d} F^c_{(0)e} R^{de:\emptyset}_{(0)} - F^b_{(0)d} F^d_{(0)e} R^{ce:\emptyset}_{(0)} + F^c_{(0)d} F^b_{(0)e} R^{de:\emptyset}_{(0)} \\ - \quad & F^c_{(0)d} F^d_{(0)e} R^{be:\emptyset}_{(0)} \Big) = \tau^{-1} \left(F^b_{(0)d} R^{cd:\emptyset}_{(0)} + F^c_{(0)d} R^{bd:\emptyset}_{(0)} \right) \quad (17)\end{aligned}$$

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- as well as the additional constraint $U^{c}_{(0)} F^{b}_{(0)c} = 0$
- The above can be rewritten as the matrix equation

$$-\frac{q}{2m} \left(\{ \mathcal{F}, \{ \mathcal{R}, \mathcal{F} \} \} \right) = \frac{1}{\tau} \left[\mathcal{F}, \mathcal{R} \right] \tag{18}$$

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- which can be solved to give

$$\mathcal{R} = \beta \left(\mathcal{F}^2 - X \Pi \right) \tag{19}$$

- Or, in matrix form

$$(\mathcal{R}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta B_z \end{pmatrix}$$

First order Equations

- We now know all zeroth order terms, hence we can substitute these into the first order equations;

$$\partial_a S_{(1)}^{a:\emptyset} = 0 \tag{20}$$

$$\partial_a S_{(1)}^{ab:\emptyset} - S_{(1)}^{\emptyset:b} = 0 \tag{21}$$

$$\partial_a S_{(1)}^{a:b} - S_{(1)}^{b:c}{}_{c} - \tau^{-1}\left(S_{(1)}^{\emptyset:b} + \frac{q}{m} F^{b}_{\langle(0)c} S^{c:\emptyset}_{(1)\rangle}\right) = 0 \tag{22}$$

$$\partial_a S_{(1)}^{abc:\emptyset} - S^{(b:c)} = 0 \tag{23}$$

$$\partial_a S_{(1)}^{ab:c} - S_{(1)}^{\emptyset:bc} - S_{(1)}^{bc:d}{}_{d} - \tau^{-1}\left(S_{(1)}^{b:c} + \frac{q}{m} F^{c}_{\langle(0)d} S^{bd:\emptyset}_{(1)\rangle}\right) = 0 \tag{24}$$

$$\partial_a S_{(1)}^{a:bc} - S_{(1)}^{(b:c)d}{}_{d} - \tau^{-1}\left(2S_{(1)}^{\emptyset:bc} + \frac{q}{m} F^{(b}_{(0)d} S^{d:c)}_{(1)}\right) = 0 \tag{25}$$

First order Equations

- If we choose F^{ab} such that, to first order in ϵ, we have only an electric wave directed along a background magnetic field, i.e.

$$\vec{B} = B_{(0)}\hat{z}$$

$$\vec{E} = \epsilon E_{(1)} e^{i(kz-\omega t)}\hat{z}$$

Solving (20) - (25) yields the following dispersion relation

$$\omega = \omega_p + \left(\frac{3}{2}\frac{k^2}{\omega_p} - \frac{3}{4}\omega_p \right)\theta - \frac{i\tau}{2}\left[\omega_p^2 - \left(2k^2 + \omega_p^2\right)\theta\right] \tag{26}$$

where ω_p is the plasma frequency and θ is the temperature of the electrons.

Future Work

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Future Work

- So far, we have found a dispersion relation for electric waves in the plasma. Next we wish to find one for electromagnetic waves.
- Our starting point was the Lorentz-Dirac equation, it may be insightful to repeat the procedure starting with the Landau-Liftshitz equation and compare results.
- We may seek alternative closure schemes to the warm fluid approximation.

Appendix - Cayley-Hamilton Theorem

- Begin by determining the characteristic equation for $\mathcal{F}$

$$det(\mathcal{F} - \lambda I) = 0$$

$$\Rightarrow \lambda^4 - \frac{1}{2} TR(\mathcal{F}^2)\lambda^2 + det(\mathcal{F}) = 0$$

- Cayley Hamilton Theorem; "Every square matrix over a commutative ring satisfies its own characteristic equation"

$$\Rightarrow \mathcal{F}^4 - \frac{1}{2} TR(\mathcal{F}^2)\mathcal{F}^2 + det(\mathcal{F}) = 0$$

$$\Rightarrow \mathcal{F}^4 = \frac{1}{2} TR(\mathcal{F}^2)\mathcal{F}^2 - det(\mathcal{F}) = 0$$

Appendix - Choosing an Ansatz for R

- It seems sensible to assume that $\mathcal{R}$ is some function of $\mathcal{F}$, and its dual $\mathcal{F}^*$
- $\mathcal{R}$ is symmetric from the definition, hence it must be made up of even powers of $\mathcal{F}$
- Powers of $\mathcal{F}$ of order 4 and higher can be eliminated using the Cayley-Hamilton Theorem
- From the property $U^a R_a{}^{b:\emptyset}$ we know it must also be orthogonal to U

$$\mathcal{R} = \alpha\Pi + \beta\mathcal{F}^2 + \gamma\Pi{\mathcal{F}^*}^2\Pi \tag{27}$$

- It can be shown that $\mathcal{F}^* = \mathcal{F}^2 - XI$, where X is the invariant ??. Hence

$$\mathcal{R} = \alpha'\Pi + \beta'\mathcal{F}^2 \tag{28}$$

- Substitute the above back into (18) and one finds