Boostrapping Gravity Solutions in 2+1 Dimensions

with Daniel Grumiller, Esperanza Lopez, Ioannis Papadimitriou and Stefan Stricker

João Aparício (IFT-UAM/CSIC)

Barcelona, October 18, 2012

Action and axi-symmetric ansatz

$$S = \frac{1}{16\pi G}\int d^3x\sqrt{-g}\left[R + 2 - \frac{1}{2}(\nabla\phi)^2 - V(\phi)\right] \quad (1)$$

Ansatz (axi-symmetric solutions):

$$ds^2 = \lambda_{\mu\nu}dx^\mu dx^\nu + \frac{e^2 d\rho^2}{-\det\lambda} \quad (2)$$

$$\lambda_{\mu\nu} = \begin{pmatrix} T+X & Y \\ Y & T-X \end{pmatrix} \quad (3)$$

$$\mathbf{X} = (T, X, Y) \quad (4)$$

$$\mathbf{X}^2 = -\det\lambda \quad (5)$$

$$S = \mathrm{Vol}\int_{\rho_0}^{\infty} d\rho e\left[\frac{e^{-2}}{2}\dot{\mathbf{X}}^2 + 2 - \frac{e^{-2}}{2}\mathbf{X}^2\dot{\phi}^2 - V(\phi)\right] \quad (6)$$

All variables depend on ρ only.

Action and axi-symmetric ansatz

$$S = \frac{1}{16\pi G}\int d^3x\sqrt{-g}\left[R + 2 - \frac{1}{2}(\nabla\phi)^2 - V(\phi)\right] \tag{1}$$

Ansatz (axi-symmetric solutions):

$$\begin{aligned}
ds^2 &= \lambda_{\mu\nu}dx^\mu dx^\nu + \frac{e^2 d\rho^2}{-\det\lambda} && (2)\\
\lambda_{\mu\nu} &= \begin{pmatrix} T+X & Y \\ Y & T-X \end{pmatrix} && (3)\\
\mathbf{X} &= (T, X, Y) && (4)\\
\mathbf{X}^2 &= -\det\lambda && (5)\\
S &= \text{Vol}\int_{\rho_0}^{\infty} d\rho e\left[\frac{e^{-2}}{2}\dot{\mathbf{X}}^2 + 2 - \frac{e^{-2}}{2}\mathbf{X}^2\dot{\phi}^2 - V(\phi)\right] && (6)
\end{aligned}$$

All variables depend on ρ only.

Equations of motion

$$\begin{aligned} \text{Einstein:} \quad & \ddot{\mathbf{X}} = -\dot{\phi}^2 \mathbf{X} \\ \text{Klein-Gordon:} \quad & \mathbf{X}^2 \ddot{\phi} + 2\mathbf{X} \cdot \dot{\mathbf{X}} \dot{\phi} = V'(\phi) \\ \text{Hamilton:} \quad & -\frac{1}{2}\dot{\mathbf{X}}^2 + 2 + \frac{1}{2}\mathbf{X}^2 \dot{\phi}^2 - V(\phi) = 0 \end{aligned} \tag{7}$$

- First-integral:

$$\epsilon_{ijk} \mathbf{X}^i \dot{\mathbf{X}}^j = \mathbf{J}_k \tag{8}$$

Equations of motion

$$\begin{aligned} \text{Einstein:} \quad & \ddot{\mathbf{X}} = -\dot{\phi}^2 \mathbf{X} \\ \text{Klein-Gordon:} \quad & \mathbf{X}^2 \ddot{\phi} + 2\mathbf{X} \cdot \dot{\mathbf{X}} \dot{\phi} = V'(\phi) \\ \text{Hamilton:} \quad & -\tfrac{1}{2}\dot{\mathbf{X}}^2 + 2 + \tfrac{1}{2}\mathbf{X}^2 \dot{\phi}^2 - V(\phi) = 0 \end{aligned} \tag{7}$$

- First-integral:

$$\epsilon_{ijk} \mathbf{X}^i \dot{\mathbf{X}}^j = \mathbf{J}_k \tag{8}$$

- The Klein-Gordon equation can be integrated. Its integration constant is fixed by the Hamilton constraint:

$$V(\rho) = 2 - \frac{1}{4} \frac{d^2}{d\rho^2} \mathbf{X}^2(\rho) \tag{9}$$

Equations of motion

$$\begin{aligned} \text{Einstein:} \quad & \ddot{\mathbf{X}} = -\dot{\phi}^2 \mathbf{X} \\ \text{Klein-Gordon:} \quad & \mathbf{X}^2 \ddot{\phi} + 2\mathbf{X} \cdot \dot{\mathbf{X}} \dot{\phi} = V'(\phi) \\ \text{Hamilton:} \quad & -\frac{1}{2}\dot{\mathbf{X}}^2 + 2 + \frac{1}{2}\mathbf{X}^2 \dot{\phi}^2 - V(\phi) = 0 \end{aligned} \tag{7}$$

- First-integral:

$$\epsilon_{ijk} \mathbf{X}^i \dot{\mathbf{X}}^j = \mathbf{J}_k \tag{8}$$

- The Klein-Gordon equation can be integrated. Its integration constant is fixed by the Hamilton constraint:

$$V(\rho) = 2 - \frac{1}{4} \frac{d^2}{d\rho^2} \mathbf{X}^2(\rho) \tag{9}$$

- The Einstein equation for $\ddot{X}^i$ depends only on X^i. This implies the following ansatz contains all the solutions

$$\mathbf{X}(\rho) = \mathbf{X}_1 f_1(\rho) + \mathbf{X}_2 f_2(\rho) \tag{10}$$

Equations of motion

$$
\begin{aligned}
\text{Einstein:} \quad & \ddot{\mathbf{X}} = -\dot{\phi}^2 \mathbf{X} \\
\text{Klein-Gordon:} \quad & \mathbf{X}^2 \ddot{\phi} + 2\mathbf{X} \cdot \dot{\mathbf{X}} \dot{\phi} = V'(\phi) \\
\text{Hamilton:} \quad & -\tfrac{1}{2}\dot{\mathbf{X}}^2 + 2 + \tfrac{1}{2}\mathbf{X}^2 \dot{\phi}^2 - V(\phi) = 0
\end{aligned}
\tag{7}
$$

- First-integral:

$$\epsilon_{ijk} \mathbf{X}^i \dot{\mathbf{X}}^j = \mathbf{J}_k \tag{8}$$

- The Klein-Gordon equation can be integrated. Its integration constant is fixed by the Hamilton constraint:

$$V(\rho) = 2 - \frac{1}{4} \frac{d^2}{d\rho^2} \mathbf{X}^2(\rho) \tag{9}$$

- The Einstein equation for $\ddot{X}^i$ depends only on X^i. This implies the following ansatz contains all the solutions

$$\mathbf{X}(\rho) = \mathbf{X}_1 f_1(\rho) + \mathbf{X}_2 f_2(\rho) \tag{10}$$

Of the SL$(2,\mathbb{R})$ use two transformations to fix $\mathbf{X}(\rho)$ as

$$\mathbf{X}(\rho) = \begin{pmatrix} T \\ X \\ Y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} f_1 - f_2 \\ -f_1 - f_2 \\ 0 \end{pmatrix} \tag{11}$$

In these variables, our equations become

$$\begin{aligned} &\text{Einstein:} && \ddot{f}_{1,2} = -\dot{\phi}^2 f_{1,2} \\ &\text{Klein-Gordon (integrated potential):} && V(\rho) = 2 - \frac{1}{2}\frac{d}{d\rho}(f_1 \dot{f}_2) \\ &\text{First-integral:} && \dot{f}_2 f_1 - \dot{f}_1 f_2 = 2j \end{aligned} \tag{12}$$

Of the $SL(2,\mathbb{R})$ use two transformations to fix $\mathbf{X}(\rho)$ as

$$\mathbf{X}(\rho) = \begin{pmatrix} T \\ X \\ Y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} f_1 - f_2 \\ -f_1 - f_2 \\ 0 \end{pmatrix} \tag{11}$$

In these variables, our equations become

$$\begin{aligned} &\text{Einstein:} & \ddot{f}_{1,2} &= -\dot{\phi}^2 f_{1,2} \\ &\text{Klein-Gordon (integrated potential):} & V(\rho) &= 2 - \tfrac{1}{2}\tfrac{d}{d\rho}(f_1 \dot{f}_2) \\ &\text{First-integral:} & \dot{f}_2 f_1 - \dot{f}_1 f_2 &= 2j \end{aligned} \tag{12}$$

Algorithm:

- Pick $f_1(\rho)$
- Obtain f_2 by integrating the first integral: $f_2 = f_1(\gamma + 2j\int^{\rho}\frac{d\rho'}{f_1^2})$
- $\dot{\phi}(\rho) = \sqrt{-\ddot{f}_1/f_1}$
- $V(\phi)$ implicitly defined to satisfy K-G equation
- Plugging $f_{1,2}$ into the line element and identify the angular and time coordinates (t,φ) as linear combinations of (x^0, x^1):
  $ds^2 = -f_2(\rho)(dx^0)^2 + f_1(\rho)(dx^1)^2 + \frac{d\rho^2}{f_1(\rho)f_2(\rho)}$

Of the $\mathrm{SL}(2,\mathbb{R})$ use two transformations to fix $\mathbf{X}(\rho)$ as

$$\mathbf{X}(\rho) = \begin{pmatrix} T \\ X \\ Y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} f_1 - f_2 \\ -f_1 - f_2 \\ 0 \end{pmatrix} \tag{11}$$

In these variables, our equations become

$$\begin{aligned} &\text{Einstein:} & \ddot{f}_{1,2} &= -\dot{\phi}^2 f_{1,2} \\ &\text{Klein-Gordon (integrated potential):} & V(\rho) &= 2 - \frac{1}{2}\frac{d}{d\rho}(f_1 \dot{f}_2) \\ &\text{First-integral:} & \dot{f}_2 f_1 - \dot{f}_1 f_2 &= 2j \end{aligned} \tag{12}$$

Algorithm:

- Pick $f_1(\rho)$
- Obtain f_2 by integrating the first integral: $f_2 = f_1(\gamma + 2j\int^{\rho}\frac{d\rho'}{f_1^2})$
- $\dot{\phi}(\rho) = \sqrt{-\ddot{f}_1/f_1}$
- $V(\phi)$ implicitly defined to satisfy K-G equation
- Plugging $f_{1,2}$ into the line element and identify the angular and time coordinates (t,φ) as linear combinations of (x^0, x^1):
  $ds^2 = -f_2(\rho)(dx^0)^2 + f_1(\rho)(dx^1)^2 + \frac{d\rho^2}{f_1(\rho)f_2(\rho)}$

Static:

$$ds^2 = -f_2(\rho)dt^2 + f_1(\rho)d\varphi^2 + \frac{d\rho^2}{f_1(\rho)f_2(\rho)} \quad (13)$$

Naturally, the qualitative properties of the solutions will depend on the choice of f_1

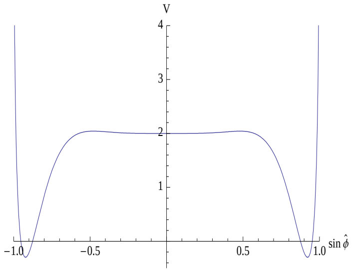
- $f_{1,2}$ linear in ρ implies asymptotically AdS space; if $f_{1,2}$ have opposite signs, the space is asymptotically dS.; f_1 linear in ρ and f_2 constant implies asymptotically flat space.
- $R = -\frac{3}{2}\dot{f}_1\dot{f}_2 - f_1\ddot{f}_2 - f_2\ddot{f}_1$. Poles may cause curvature singularities at $\rho = \rho_s$. This maybe be acceptable if the space also contains a Killing horizon or a center at $\rho_0 > \rho_s$.
- $f_2 = f_1(\gamma + 2j\int^{\rho}\frac{d\rho'}{f_1^2})$. If f_2 has a zero at some ρ_0 and f_1 doesn't, we have a Kiling horizon.
- If f_1 has a zero at some ρ_0 and f_2 doesn't, we have a center. Furthermore we demand $f_2\dot{f}_1^2\Big|_{\rho=\rho_0} = 4$.
- If both $f_{1,2}$ have a zero at the same point, the horizon is a Poincaré patch horizon.

Example: Asymptotically flat black holes with scalar hair

Remember: $ds^2 = -f_1(\rho)dt^2 + f_2(\rho)dx^2 + \frac{d\rho^2}{f_1(\rho)f_2(\rho)}$

Solution:

- $f_1 = 1 - \frac{1}{\rho}, \qquad \rho \in (1, \infty), \qquad \dot{f}_2 f_1 - \dot{f}_1 f_2 = 2j$
- $f_2 = f_1 \left(A - 2j\left[\frac{1}{\rho-1} - \rho - 2\log(\rho - 1)\right]\right) \simeq 2j\rho + \cdots$
- $R = \frac{5j}{\rho^2} + \cdots$
- $\phi = -\sqrt{2}\pi + 2\sqrt{2}\arctan\sqrt{\frac{\rho}{\Omega} - 1} \simeq -2\sqrt{2}\sqrt{\frac{\Omega}{\rho}} + \cdots, \qquad \phi \in (-\pi\sqrt{2}, 0)$
- $V(\phi) = 2 + jx^4 + x^6(A - 4j + 4j\log\frac{1-x^2}{x^2}) - \frac{3}{2}x^8(A + 2j + 4j\log\frac{1-x^2}{x^2}), x = \sin\frac{\phi}{2\sqrt{2}}$



Can we find a family of solutions?

$$\begin{aligned} &\text{Einstein:} && \ddot{\mathbf{X}} = -\dot{\phi}^2\mathbf{X} && \rightarrow +6 \\ &\text{Klein-Gordon:} && \mathbf{X}^2\ddot{\phi} + 2\mathbf{X}\cdot\dot{\mathbf{X}}\dot{\phi} = V'(\phi) && \rightarrow +2 \\ &\text{Hamilton:} && -\tfrac{1}{2}\dot{\mathbf{X}}^2 + 2 + \tfrac{1}{2}\mathbf{X}^2\dot{\phi}^2 - V(\phi) = 0 && \rightarrow -1 \end{aligned} \tag{14}$$

Can we find a family of solutions?

$$\begin{array}{lcr} \text{Einstein:} & \ddot{\mathbf{X}} = -\dot{\phi}^2\mathbf{X} & \rightarrow +6 \\ \text{Klein-Gordon:} & \mathbf{X}^2\ddot{\phi} + 2\mathbf{X}\cdot\dot{\mathbf{X}}\dot{\phi} = V'(\phi) & \rightarrow +2 \\ \text{Hamilton:} & -\frac{1}{2}\dot{\mathbf{X}}^2 + 2 + \frac{1}{2}\mathbf{X}^2\dot{\phi}^2 - V(\phi) = 0 & \rightarrow -1 \end{array} \tag{14}$$

We used two transformations of $SL(2,\mathbb{R})$ to write

$$\mathbf{X}(\rho) = \frac{1}{2}\begin{pmatrix} f_1 - f_2 \\ -f_1 - f_2 \\ 0 \end{pmatrix} \qquad \rightarrow -2 \tag{15}$$

Of the remaining **five** integration constants, **three** are gravity

- Lorentz boosts $f_1 \rightarrow \gamma f_1$, $f_2 \rightarrow f_2/\gamma$
- Shifts $\rho \rightarrow \rho + \rho_0$
- Particle angular momentum j

and the other **two** are the scalar leading and subleading modes $\phi_\pm$.
We have a total of **five** parameters: $\gamma, \rho_0, j, \phi_\pm$.

Can we find a family of solutions?

$$\begin{array}{lcl} \text{Einstein:} & \ddot{\mathbf{X}} = -\dot{\phi}^2 \mathbf{X} & \rightarrow +6 \\ \text{Klein-Gordon:} & \mathbf{X}^2 \ddot{\phi} + 2\mathbf{X} \cdot \dot{\mathbf{X}} \dot{\phi} = V'(\phi) & \rightarrow +2 \\ \text{Hamilton:} & -\frac{1}{2}\dot{\mathbf{X}}^2 + 2 + \frac{1}{2}\mathbf{X}^2 \dot{\phi}^2 - V(\phi) = 0 & \rightarrow -1 \end{array} \tag{14}$$

We used two transformations of $SL(2, \mathbb{R})$ to write

$$\mathbf{X}(\rho) = \frac{1}{2} \begin{pmatrix} f_1 - f_2 \\ -f_1 - f_2 \\ 0 \end{pmatrix} \qquad \rightarrow -2 \tag{15}$$

Of the remaining **five** integration constants, **three** are gravity

- Lorentz boosts $f_1 \rightarrow \gamma f_1,\ f_2 \rightarrow f_2/\gamma$
- Shifts $\rho \rightarrow \rho + \rho_0$
- Particle angular momentum j

and the other **two** are the scalar leading and subleading modes $\phi_\pm$.
We have a total of **five** parameters: $\gamma, \rho_0, j, \phi_\pm$.

Family of solutions

Rescalings!

$$\rho \rightarrow \Omega\rho \quad f_{1,2} \rightarrow \Omega f_{1,2} \quad j \rightarrow \Omega j \tag{16}$$

$$\frac{d}{d\rho} \rightarrow \frac{1}{\Omega}\frac{d}{d\rho} \qquad \phi \rightarrow \phi \tag{17}$$

$$\dot{f}_{1,2} \rightarrow \dot{f}_{1,2} \qquad \ddot{f}_{1,2} \rightarrow \frac{1}{\Omega}\ddot{f}_{1,2} \tag{18}$$

$$\begin{aligned} &\text{Einstein:} & \ddot{f}_{1,2} &= -\dot{\phi}^2 f_{1,2} \\ &\text{Klein-Gordon (integrated potential):} & V(\rho) &= 2 - \frac{1}{2}\frac{d}{d\rho}(f_1\dot{f}_2) \\ &\text{First-integral:} & \dot{f}_2 f_1 - \dot{f}_1 f_2 &= 2j \end{aligned} \tag{19}$$

Example: Asymptotically flat black holes - family of solutions

Remember: $ds^2 = -f_2(\rho)dt^2 + f_1(\rho)dx^2 + \frac{d\rho^2}{f_1(\rho)f_2(\rho)}$

Solution:

- $f_1 = 1 - \frac{\Omega}{\rho}, \qquad \Omega > 0, \qquad \rho \in (\rho_h, \infty), \qquad f_2(\rho_h) = 0, \qquad \rho_h > \Omega$
- $f_2 = \Omega^2 f_1 \left(A - 2\xi \left[\frac{\Omega}{\rho - \Omega} - \frac{\rho}{\Omega} - 2\log(\frac{\rho}{\Omega} - 1) \right] \right) \qquad \dot{f}_2 f_1 - \dot{f}_1 f_2 = 2\xi\Omega$
- $R = 5\xi \frac{\Omega^2}{\rho^2} + \cdots$
- $\phi = -\sqrt{2}\pi + 2\sqrt{2}\arctan\sqrt{\frac{\rho}{\Omega} - 1} \simeq -2\sqrt{2}\sqrt{\frac{\Omega}{\rho}} + \cdots, \qquad \phi \in (-\pi\sqrt{2}, 0)$
- $V(\phi) = 2 + \xi x^4 + \left(A - 4\xi + 4\xi \log\frac{1-x^2}{x^2} \right) x^6 - \frac{3}{2}\left(A + 2\xi + 4\xi \log\frac{1-x^2}{x^2} \right) x^8$
  $x = \sin(\phi/\sqrt{8})$

Asymptotics

Brown-Henneaux asymptotics:

$$f_{1,2} = 2\rho + B_{1,2} + \mathcal{O}(\rho^{-\delta_{1,2}}), \qquad 1 \leq \delta_{1,2} > 0 \tag{20}$$

Simple generalization of Brown-Henneaux asymptotics:

$$f_1 = 2\rho + A\Omega^{1-\epsilon}\rho^{\epsilon} + B\Omega \log\rho + \rho_0 + \cdots \tag{21}$$

$$\phi = \phi_- \rho^{-\Delta_-/2} + \phi_+ \rho^{-\Delta_+/2} + \cdots, \qquad \Delta_\pm = 1 \pm \epsilon, \qquad m^2 = \epsilon^2 - 1 \tag{22}$$

Holographic renormalization and thermodynamics

$$\begin{aligned}
\Gamma_+ &= \int d^3x\sqrt{-g}\left[R+2-\frac{1}{2}(\partial\phi)^2-V(\phi)\right]+2\int d^2x\sqrt{-\gamma}\,[K-U(\phi)] \\
S_{\mathcal{W}} &= -4\mathrm{Vol}\epsilon\phi_-\phi_+-2\mathrm{Vol}(\mathcal{W}(\phi_-)-\phi_-\mathcal{W}'(\phi_-)) \quad (23) \\
U(\phi) &= 1+\frac{1-\epsilon}{4}\phi^2+\cdots \quad (24) \\
& \begin{array}{ll} \text{Dirichlet} & \mathcal{J}=\phi_- \\ \text{Neumann} & \mathcal{J}=2\epsilon\phi_+ \\ \text{Mixed} & \mathcal{J}=2\epsilon\phi_+-\mathcal{W}'(\phi_-) \end{array} \quad (25) \\
\langle T_{ij}\rangle_+ &= -2\lim_{\rho\to\infty}\left[K_{ij}-(K-U(\phi))\gamma_{ij}\right] \quad (26) \\
\langle T_{ij}\rangle_- &= \langle T_{ij}\rangle_+-(\mathcal{W}(\phi_-)+\phi_-\mathcal{J})g_{(0)\mu\nu} \quad (27)
\end{aligned}$$

Also, thermodynamics: temperature, mass, angular momentum and entropy

Solutions

- Reproduced results from the literature: axi-symmetric solutions
- Brown-Henneaux solitons
- Black holes (BH and non-BH)
- Solutions with exponential potential (neither flat nor AdS)